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On the Convergence and Iterates of q-Bernstein Polynomials

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The convergence properties of q-Bernstein polynomials are investigated. When $q \geqslant 1$ is fixed the generalized Bernstein polynomials $\mathscr{B}_n f$ of f, a one parameter family of Bernstein polynomials, converge to f as $n \to \infty$ if f is a polynomial. It is proved that, if the parameter $0 < q < 1$ is fixed, then $\mathscr{B}_n f \to f$ if and only if f is linear. The iterates of $\mathscr{B}_n f$ are also considered. It is shown that $\mathscr{B}_n^M f$ converges to the linear interpolating polynomial for f at the endpoints of $[0, 1]$, for any fixed $q > 0$, as the number of iterates $M \to \infty$. Moreover, the iterates of the Boolean sum of $\mathscr{B}_n f$ converge to the interpolating polynomial for f at $n + 1$ geometrically spaced nodes on $[0, 1]$.

Key Words: q-Bernstein polynomials; Stirling polynomials; iterates of the q-Bernstein operator; interpolation.

1. INTRODUCTION

It is well known that the Bernstein polynomials, defined by

$$B_n(f;x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right)\binom{n}{r} x^r (1-x)^{n-r} \tag{1.1}$$

converge to $f(x)$ when $f \in C[0, 1]$. Phillips [16] generalized (1.1) to give

$$\mathscr{B}_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \qquad 0 \leqslant x \leqslant 1, \tag{1.2}$$

where an empty product denotes 1 and $f_r = f([r]/[n])$, where

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1, \end{cases}$$

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and the q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$ is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n].[n-1]\cdots[n-r+1]}{[r].[r-1]\cdots[1]}$$

for $n \geqslant r \geqslant 1$, having the value 1 when $r = 0$, and the value zero otherwise. The q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$ (see [1]) satisfies Pascal-type identities, one of which we will use later, and is the generating function for counting restricted partitions. Note that (1.2) reduces to (1.1) when $q = 1$. Polynomials (1.2) nicely generalize many properties of the classical Bernstein polynomials (1.1). The q-Bernstein polynomial (1.2) may be written in the q-difference form (see [16])

$$\mathscr{B}_n(f; x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r, \tag{1.3}$$

where $\Delta^r f_i = \Delta^{r-1} f_{i+1} - q^{r-1} \Delta^{r-1} f_i$ for $r \geqslant 1$ and $\Delta^0 f_i = f_i$. It is easily verified that

$$\Delta^r f_i = \sum_{k=0}^{r} (-1)^k q^{k(k-1)/2} \begin{bmatrix} r \\ k \end{bmatrix} f_{r+i-k}. \tag{1.4}$$

We may deduce from (1.3) that $\mathscr{B}_n$ reproduces linear polynomials, that is,

$$\mathscr{B}_n(ax + b; x) = ax + b, \qquad a, b \in \mathbb{R}.$$

It follows directly from (1.2) that, for any $0 < q \leqslant 1$, $\mathscr{B}_n$ is a monotone linear operator which maps positive continuous functions on $[0, 1]$ to positive continuous functions on $[0, 1]$. It also follows from (1.3) that

$$\mathscr{B}_n(x^2; x) = x^2 + \frac{x(1-x)}{[n]}. \tag{1.5}$$

Thus $\mathscr{B}_n(x^2; x) \to x^2$ as $n \to \infty$ if and only if $q \geqslant 1$. If we (here only) define $[n] = 1 + q_n + \cdots + q_n^{n-1}$, so that the q-integer $[n]$ is given in terms of a value of q which depends on the degree n in (1.2), then, taking a sequence $q = q_n$, with $0 < q_n \leqslant 1$, such that $[n] \to \infty$ as $n \to \infty$, it follows that $\mathscr{B}_n(x^2; x) \to x^2$. Thus, by using the Bohman–Korovkin theorem, the generalized Bernstein polynomials $\mathscr{B}_n f$ converge to f for all $f \in C[0, 1]$ (see [16]). A discussion on a Voronovskaya-type theorem for the rate of convergence can also be found in [16]. In Section 2 of the present paper we will prove that for a fixed $q \geqslant 1$, $\mathscr{B}_n f \to f$ as $n \to \infty$ if f is a polynomial. Moreover, for a fixed q, $0 < q < 1$, we will have the uniform convergence $\mathscr{B}_n f \to f$ if and only if f is linear.

The convergence of the iterates and Boolean sum of (1.2) will be discussed in Section 3.

The q-Bernstein polynomial shares the well-known shape-preserving properties of the classical Bernstein polynomial. For example, when the function f is convex then $\mathscr{B}_{n-1}(f;x) \geqslant \mathscr{B}_n(f;x)$ for $n \geqslant 2$ and any $0 < q \leqslant 1$ (see [12]). As a consequence of this one can show that the approximation to a convex function by q-Bernstein polynomials is one sided, with $\mathscr{B}_n f \geqslant f$ for all n (see [14]). In addition, $\mathscr{B}_n f$ behaves in a very nice way when we vary the parameter q: it is proved in [6] that $\mathscr{B}_n^r(f;x) \leqslant \mathscr{B}_n^q(f;x)$ for any $0 < q \leqslant r \leqslant 1$. It is also shown in [6] that monotonic and convex functions result monotonic and convex q-Bernstein polynomials, respectively.

In CAGD applications the choice of the basis used for designing parametric curves and surfaces is important. Normalized totally positive bases are most suitable for this purpose. The basis functions used in (1.2), namely

$$\begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \qquad 0 \leqslant r \leqslant n, \quad x \in [0,1]$$

form a normalized totally positive basis (see [6, 14]). The special case of this, where $q = 1$, gives the normalized totally positive basis used in (1.1). Note that a system of functions $\{\Phi_0, \Phi_1, \ldots, \Phi_n\}$ is called totally positive if all its collocation matrices $(\Phi_j(x_i))_{i,j=0}^n$ are totally positive, that is, all their minors are nonnegative. In addition, if $\{\Phi_0, \Phi_1, \ldots, \Phi_n\}$ are linearly independent and positive such that $\sum_{i=0}^n \Phi_i = 1$ then $\{\Phi_0, \Phi_1, \ldots, \Phi_n\}$ is a normalized totally positive basis. They also possess a variation-diminishing property. This means that, for any vector $v_0, \ldots, v_n \in \mathbb{R}^{n+1}$ the number of strict sign changes of $\sum_{i=0}^n v_i \Phi_i$ is less than or equal to the number of strict sign changes in the sequence $v_0, \ldots, v_n$. For more information on this subject see [2, 7] and the references therein.

The de Casteljau algorithm is fundamental in the application of curve and surface design. Phillips [15] established a generalization of that algorithm. Given $b_0^{[0]}, b_1^{[0]}, \ldots, b_n^{[0]} \in \mathbb{R}^2$, where $b_i^{[0]} = ([i]/[n], f_i)$ for $i = 0, 1, \ldots, n$, we set

$$b_r^{[m]} := (q^r - q^{m-1}x) b_r^{[m-1]} + x b_{r+1}^{[m-1]} \begin{cases} m = 1, 2, \ldots, n, \\ r = 0, 1, \ldots, n-m. \end{cases}$$

Then, $b_0^{[n]}$ evaluates the q-Bernstein polynomial (1.2) and gives (1.1) as a special case when $q = 1$.

2. CONVERGENCE

Throughout the paper convergence means *uniform* convergence on the interval [0, 1]. The following representation of the q-Bernstein polynomial of a monomial is obtained in [6]. It involves Stirling polynomials and leads to some results on convergence. For any fixed integer i, the q-Bernstein polynomials of monomials can be written explicitly as

$$\mathcal{B}_n(x^i; x) = \sum_{j=0}^{i} \lambda_j [n]^{j-i} S_q(i,j) x^j, \tag{2.1}$$

where

$$\lambda_j = \prod_{r=0}^{j-1} \left(1 - \frac{[r]}{[n]}\right),$$

an empty product denotes 1, and

$$S_q(i,j) = \frac{1}{[j]! q^{j(j-1)/2}} \sum_{r=0}^{j} (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix} [j-r]^i, \qquad 0 \leqslant j \leqslant i. \tag{2.2}$$

The polynomials $S_q(i,j)$ are also given by the generating function

$$x^i = \sum_{j=0}^{i} S_q(i,j) x_j(x), \tag{2.3}$$

where $x_j(x) = x(x-[1])(x-[2]) \cdots (x-[j-1])$. One may verify either by induction on i, using

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r \end{bmatrix} + q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix},$$

or the generating function above that

$$S_q(i+1,j) = S_q(i,j-1) + [j] S_q(i,j) \tag{2.4}$$

with $S_q(0,0) = 1$, $S_q(i,0) = 0$ for $i > 0$, and we define $S_q(i,j) = 0$ for $j > i$. (Note that this last property ensures that (2.1) holds for all n.) We call the $S_q(i,j)$ the Stirling polynomials of the second kind since when $q = 1$ they are the Stirling numbers of the second kind. There are many interesting properties of Stirling polynomials in combinatorics (see for example [10]).

THEOREM 2.1. *Let $q \geqslant 1$ be a fixed real number. Then, for any polynomial p,*

$$\lim_{n\to\infty} \mathscr{B}_n(p;x) = p(x).$$

Proof. Let

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m.$$

Then, for $n > m$, we may write

$$\mathscr{B}_n(p;x) = \mathbf{a}^T \mathbf{A}\mathbf{x}, \tag{2.5}$$

where $\mathbf{a}$ is the vector whose elements are the coefficients of p, $\mathbf{A}$ is an $(m+1)\times(m+1)$ lower triangular matrix with the elements

$$a_{i,j} = \begin{cases} \lambda_j [n]^{j-i} S_q(i,j), & 0 \leqslant j \leqslant i, \\ 0, & i < j, \end{cases} \tag{2.6}$$

and $\mathbf{x}$ is the vector whose elements form the standard basis for the space of polynomials P_m of degree m. When $q \geqslant 1$ it is easily seen that

$$\frac{1}{[n]} \to 0 \qquad \text{and} \qquad \lambda_j \to 1$$

for all j as $n \to \infty$. Hence all entries of $\mathbf{A}$ except its diagonal converge to zero. Further, it is clear from the fact that $S_q(i,j) = 1$ when $i = j$, every element of the diagonal of $\mathbf{A}$ converges to unity. Thus the matrix $\mathbf{A}$ tends to the $(m+1)\times(m+1)$ identity matrix. This completes the proof. ∎

LEMMA 2.1. *Let $0 < q < 1$ be a fixed real number. Then*

$$\lim_{n\to\infty} \mathscr{B}_n(p;x) = p(x)$$

if and only if $p(x)$ is linear.

Proof. We only require to prove the converse, since the q-Bernstein operator reproduces linear functions. Let $p(x) \in P_m$ where $m \geqslant 2$. We may represent $\mathscr{B}_n(p;x)$ as in (2.5) and (2.6). When $0 < q < 1$ is fixed,

$$\frac{1}{[n]} \to 1 - q, \quad \lambda_j \to q^{j(j-1)/2}, \qquad n \to \infty.$$

Thus, the matrix $\mathbf{A}$ does not converge to the identity matrix but to the matrix whose elements are

$$a_{i,j} = \begin{cases} q^{j(j-1)/2}(1-q)^{i-j} S_q(i,j), & 0 \leqslant j \leqslant i, \\ 0, & i < j. \end{cases}$$

Hence $\lim_{n\to\infty} \mathscr{B}_n(p; x)$ does not converge to p unless p is linear. ∎

THEOREM 2.2. *Let $0 < q < 1$ be a fixed real number and $f \in C[0,1]$. Then*

$$\lim_{n\to\infty} \mathscr{B}_n(f; x) = f(x)$$

if and only if $f(x)$ is linear.

Proof. It is enough to show that linearity of f is necessary for the uniform convergence of $\mathscr{B}_n f$. We choose a polynomial $p(x)$ satisfying

$$|p(x) - f(x)| < \varepsilon, \qquad 0 \leqslant x \leqslant 1,$$

for a given $\varepsilon > 0$. Since $\mathscr{B}_n$ is a monotone linear operator for $0 < q < 1$ we obtain

$$|\mathscr{B}_n(p; x) - \mathscr{B}_n(f; x)| < \mathscr{B}_n(\varepsilon; x) = \varepsilon.$$

By the above assumption, $\mathscr{B}_n(f; x) \to f(x)$ uniformly on $[0,1]$. On using the lemma above, $\mathscr{B}_n(p; x) \to a_0 + a_1 x$. Thus,

$$|(a_0 + a_1 x) - f(x)| < \varepsilon, \qquad 0 \leqslant x \leqslant 1. \quad ∎$$

3. THE ITERATES

The iterates of the q-Bernstein polynomial are defined by

$$\mathscr{B}_n^{M+1}(f; x) = \mathscr{B}_n(\mathscr{B}_n^M(f; x); x), \qquad M = 1, 2, \ldots, \tag{3.1}$$

where $\mathscr{B}_n^1(f; x) = \mathscr{B}_n(f; x)$. We will investigate the convergence properties of the iterates as $M \to \infty$. For the classical Bernstein polynomials, the iterates converge to linear end point interpolation on $[0,1]$. Kelisky and Rivlin [9] considered this problem both when M is independent of the degree of $B_n f$ and when M is dependent on n. Several generalizations of this problem have been studied. Micchelli [11] introduced certain linear combinations of the Bernstein polynomials. These linear combinations, which may be regarded as Boolean sums, are discussed in [5, 17, 18]. They proved that the iterated

Boolean sum of (1.1) converges to the interpolating polynomial of f of degree n at equally spaced points on $[0, 1]$. Wenz [18] also obtained similar results for the Bernstein–Schoenberg and Sablonniére operators as well as for the Bernstein operator and Bernstein–Durmeyer operator over triangles. Cooper and Waldron [3] investigated the eigenstructure of the Bernstein operator B_n, and applied it to iterates of the Bernstein operator.

Theorem 3.1. *Let $q > 0$ be a fixed real number. Then*

$$\lim_{M\to\infty} \mathscr{B}_n^M(f;x) = f(0) + (f(1) - f(0))x. \tag{3.2}$$

Proof. On using the q-difference form of the q-Bernstein polynomials we obtain, on applying the q-Bernstein operator twice,

$$\mathscr{B}_n^2(f;x) = \mathbf{f}^T\,\mathbf{A}\mathbf{x}, \tag{3.3}$$

where the vectors $\mathbf{f}$ and $\mathbf{x}$ are

$$\mathbf{f} = \left[\begin{bmatrix} n \\ 0 \end{bmatrix}\Delta^0 f_0, \begin{bmatrix} n \\ 2 \end{bmatrix}\Delta^1 f_0, \ldots, \begin{bmatrix} n \\ n \end{bmatrix}\Delta^n f_0\right]^T, \qquad \mathbf{x} = [1, x, \ldots, x^n]^T \tag{3.4}$$

and the matrix $\mathbf{A}$ is an $(n+1)\times(n+1)$ lower triangular matrix as defined in (2.6), with m replaced by n. The eigenvalues of $\mathbf{A}$ are $\lambda_0, \lambda_1, \ldots, \lambda_n$ and satisfy

$$1 = \lambda_0 = \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0. \tag{3.5}$$

The matrix $\mathbf{A}$ is diagonalizable: there exists a diagonal matrix $\mathbf{D}$ and a matrix $\mathbf{P}$ such that $\mathbf{AP} = \mathbf{PD}$. Here $\mathbf{D}$ denotes the $(n+1)\times(n+1)$ diagonal matrix whose elements are the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n$ and $\mathbf{P}$ is an $(n+1)\times(n+1)$ lower triangular matrix whose column vectors are the eigenvectors of $\mathbf{A}$. The matrix $\mathbf{P}$ can be normalized so that the entries on its main diagonal are all 1. Since the q-Bernstein polynomials interpolate the function at the end points, $\mathscr{B}_n(x^i; 1) = 1$ for $i = 0, 1, \ldots, n$ and it follows that $\mathbf{A}$ is a stochastic matrix, that is, its row sums are all 1, since

$$\sum_{j=0}^{i} \lambda_j [n]^{j-i} S_q(i,j) = 1, \qquad i = 0, 1, \ldots, n.$$

We note that stochastic matrices are used in the study of Markov chains, including applications to population migration models, since each row of a stochastic matrix may be thought of as a discrete probability distribution on a sample space.

It follows from $\mathbf{A} = \mathbf{PDP}^{-1}$ and $\mathbf{AP} = \mathbf{PD}$ that

$$p_{0,0} = 1, \quad p_{i,0} = 0, \qquad \text{for } i = 1, 2, \ldots, n$$

and

$$\sum_{j=1}^{i} a_{i,j} p_{j,1} = p_{i,1}, \qquad \text{for } i = 1, 2, \ldots, n.$$

We deduce from the latter equation that

$$p_{i,1} = 1, \qquad \text{for } i = 1, 2, \ldots, n.$$

It will be enough to know the first column and second row of lower triangular matrix $\mathbf{P}^{-1}$. We calculate from $\mathbf{P}^{-1}\mathbf{A} = \mathbf{DP}^{-1}$ that the first column of $\mathbf{P}^{-1}$ is $[1, \mathbf{0}]^T$ and the second row is $[0, 1, \mathbf{0}]$, where $\mathbf{0}$ denotes an appropriate zero vector. Now, we obtain from (3.1) and (3.3) that

$$\mathscr{B}_n^M(f; x) = \begin{bmatrix} n \\ 0 \end{bmatrix} \Delta^0 f_0 + \begin{bmatrix} n \\ 1 \end{bmatrix} \Delta^1 f_0 x + \mathscr{B}_n^{M-1}(x^2; x) + \cdots$$

$$+ \begin{bmatrix} n \\ n \end{bmatrix} \Delta^n f_0 \mathscr{B}_n^{M-1}(x^n; x)$$

so that

$$\mathscr{B}_n^M(f; x) = \mathbf{f}^T \mathbf{A}^{M-1} \mathbf{x} = \mathbf{f}^T \mathbf{PD}^{M-1} \mathbf{P}^{-1} \mathbf{x}. \tag{3.6}$$

This implies that $\mathscr{B}_n^M(f; x)$ converges if and only if $\mathbf{A}^{M-1}$ converges. Since $\mathbf{P}$ and $\mathbf{P}^{-1}$ are triangular matrices and have elements as calculated above, and

$$\lim_{M \to \infty} \lambda_i^{M-1} = 1, \quad i = 0, 1 \qquad \text{and} \qquad \lim_{M \to \infty} \lambda_i^{M-1} = 0, \;\; i = 2, 3, \ldots, n,$$

we obtain

$$\lim_{M \to \infty} \mathbf{PD}^{M-1} \mathbf{P}^{-1} = \mathbf{C},$$

where $\mathbf{C}$ is the $(n+1) \times (n+1)$ matrix with elements

$$c_{i,j} = \begin{cases} 1, & i = j = 0 \text{ or } j = 1, \;\; i \geqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, from (3.6) we have

$$\lim_{M\to\infty} \mathbf{f}^T \mathbf{A}^{M-1}\mathbf{x} = \mathbf{f}^T \mathbf{C}\mathbf{x}.$$

The latter equation gives

$$\lim_{M\to\infty} \mathscr{B}_n^M(f;x) = f_0 + x\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \Delta^i f_0.$$

We obtain from (1.3), on putting $x = 1$, that

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \Delta^i f_0 = f_n - f_0.$$

We conclude that the iterates of q-Bernstein polynomials converge to the linear end-point interpolating polynomial on [0, 1]. ∎

Next, we will investigate the Boolean sum of q-Bernstein polynomials. First, it is necessary to introduce some notation. The Boolean sum of two operators $\mathscr{A}$ and $\mathscr{B}$ is defined by

$$\mathscr{A} \oplus \mathscr{B} = \mathscr{A} + \mathscr{B} - \mathscr{A}\circ\mathscr{B}.$$

Let $\oplus^0\mathscr{B} = \mathscr{I}$ be the identity operator and $\oplus^1\mathscr{B} = \mathscr{B}$. The iterated Boolean sum of $\mathscr{B}$ is defined recursively by

$$\oplus^{M+1}\mathscr{B} = \mathscr{B} \oplus (\oplus^M\mathscr{B}), \qquad M \geqslant 1.$$

The following two lemmas will be useful, since the Boolean sum of a linear operator $\mathscr{B}$ has a connection with the Neumann series form of its matrix representation.

LEMMA 3.1. *Let* $\mathbf{A}$ *be an* $(n+1)\times(n+1)$ *matrix whose elements are defined by the equation* (2.6). *Then its inverse* $\mathbf{B}$ *is the convergent Neumann series*

$$\mathbf{I} + \sum_{j=1}^{\infty} (\mathbf{I} - \mathbf{A})^j = \mathbf{B} \tag{3.7}$$

and is given explicitly by

$$b_{i,j} = \begin{cases} \dfrac{1}{\lambda_i}[n]^{j-i} s_q(i,j), & 0 \leqslant j \leqslant i \leqslant n, \\ 0, & i < j, \end{cases} \tag{3.8}$$

where $s_q(i,j)$ *denotes a Stirling polynomial of the first kind.*

Proof. Note that the eigenvalues of **A** are all less or equal to 1. Hence it is easily seen that $\rho(\mathbf{I}-\mathbf{A})<1$, where $\rho(\mathbf{A})$ denotes the spectral radius of **A**. This implies that the series on the left of (3.7) is convergent. Next, we define the generating function for the Stirling polynomials of the first kind, which are q-analogues of Stirling numbers. The Stirling polynomials of the first kind are given by

$$x_i(x) = \sum_{j=0}^{i} s_q(i,j)x^j. \tag{3.9}$$

We set $s_q(0,0)=1$, $s_q(i,0)=0$ for $i>0$ and $s_q(i,j)=0$ for $j>i$. Note that, for $0\leqslant i,j\leqslant n$,

$$\sum_{k=0}^{n} s_q(i,k)S_q(k,j) = \sum_{k=0}^{n} S_q(i,k)s_q(k,j) = \delta_{i,j}, \tag{3.10}$$

where $\delta_{i,j}$ is the Kronecker delta function. Now it can be easily verified from (3.8), (3.10) and (2.6) that $\mathbf{AB}=\mathbf{I}$. ∎

We note that the above matrix **A** can be obtained from the Vandermonde matrix $\mathbf{V}=\mathbf{V}(x_0,x_1,\ldots,x_n)$. A triangular factorization which involves complete symmetric functions and a bidiagonal factorization of **V** is given explicitly in [13]. Another factorization, in the form $\mathbf{V}=\mathbf{LDU}$, is given by Gohberg and Koltracht [4].

The upper triangular matrix **U** in $\mathbf{V}=\mathbf{LU}$ has the elements

$$u_{i,j} = \tau_{j-i}(x_0,\ldots,x_i)\prod_{t=0}^{i-1}(x_i-x_t), \qquad 0\leqslant i\leqslant j\leqslant n,$$

with an empty product denoting 1, where $\tau_r(x_0,\ldots,x_i)$ is the rth complete symmetric function in the variables $x_0,\ldots,x_i$. It follows from

$$\tau_{i-j}(x_0,\ldots,x_j) = f[x_0,\ldots,x_j], \quad f(x)=x^i, \qquad 0\leqslant j\leqslant i\leqslant n,$$

where $f[x_0,\ldots,x_j]$ denotes a divided difference, and (2.2) that, on putting $x_j=[j]/[n]$, we obtain

$$\tau_{i-j}(x_0,\ldots,x_j) = [n]^{j-i}S_q(i,j).$$

Thus, the transpose of **U** has elements

$$\mathbf{U}^T = u_{i,j} = [n]^{j-i}S_q(i,j)[j]! \qquad 0\leqslant j\leqslant i\leqslant n. \tag{3.11}$$

Therefore, we may write the above matrix $\mathbf{A}$ as a product of $\mathbf{A} = \mathbf{U}^T\tilde{\mathbf{D}}$ where $\tilde{\mathbf{D}}$ is a totally positive diagonal matrix having the elements $\lambda_j/[j]!$, where $[j]!$ denotes the product $[j][j-1]\cdots[1]$. The Vandermonde matrix and its triangular factors are totally positive matrices for $0<x_0<x_1<\cdots<x_n$. See [13]. Thus, we deduce that the matrix $\mathbf{U}^T$ is a totally positive matrix and that $\mathbf{A}$ is also totally positive since it is written as a product of totally positive matrices.

LEMMA 3.2. *Let $L_n f$ denote the interpolating polynomial for the function f at the $n+1$ geometrically spaced nodes, $[i]/[n]$, $i=0,1,\dots,n$, on $[0,1]$. Then*

$$L_n f = \sum_{i=0}^{n} \frac{1}{q^{i(i-1)/2}\,[i]!}\Delta^i f_0 \sum_{j=0}^{i} [n]^{j} s_q(i,j)x^j. \tag{3.12}$$

Proof. We write the divided difference form of the interpolating polynomial for f at the points $x_i = [i]/[n]$, $i=0,1,\dots,n$, in the form

$$L_n f = \sum_{i=0}^{n} \pi_i f[x_0,x_1,\dots,x_i], \tag{3.13}$$

where

$$\pi_i(x) = x(x-[1]/[n])(x-[2]/[n])\cdots(x-[i-1]/[n]), \tag{3.14}$$

with $\pi_0(x)=1$. It can be shown by induction on i (see [16]) that

$$f[x_0,x_1,\dots,x_i] = \frac{[n]^i}{q^{i(i-1)/2}\,[i]!}\Delta^i f_0. \tag{3.15}$$

On using (3.9) and (3.14) we see that

$$\pi_i(x) = \sum_{j=0}^{i} [n]^{j-i} s_q(i,j)x^j.$$

The proof follows from the latter equation and (3.13), and (3.15). ∎

THEOREM 3.2. *The iterated Boolean sum of the q-Bernstein operator $\oplus^M \mathcal{B}_n(f;x)$ associated with the function $f(x)\in C[0,1]$ converges to the interpolating polynomial $L_n f$ of degree n of $f(x)$ at the points $x_i=[i]/[n]$, $i=0,1,\dots,n$.*

Proof. It follows from the definition that the second iterated Boolean sum of the q-Bernstein operator $\mathcal{B}_n$ satisfies

$$\oplus^2\mathcal{B}_n = \mathcal{B}_n + \mathcal{B}_n - \mathcal{B}_n(\mathcal{B}_n) = \mathcal{B}_n(\mathcal{I} + (\mathcal{I} - \mathcal{B}_n)).$$

The second iteration of $\mathscr{B}_n$ associated with f may be written in the matrix form (see Eq. (3.3))

$$\oplus^2 \mathscr{B}_n(f;x) = \mathbf{f}^T(\mathbf{I} + (\mathbf{I} - \mathbf{A}))\mathbf{x}.$$

One may prove by induction on M, using

$$\mathscr{B}_n(\mathscr{I} + (\mathscr{I} - \mathscr{B}_n) + \cdots + (\mathscr{I} - \mathscr{B}_n)^{M-1}) = \mathscr{I} - (\mathscr{I} - \mathscr{B}_n)^M,$$

that

$$\oplus^M \mathscr{B}_n = \mathscr{B}_n(\mathscr{I} + (\mathscr{I} - \mathscr{B}_n) + \cdots + (\mathscr{I} - \mathscr{B}_n)^{M-1}). \qquad (3.16)$$

Thus, (3.16) may be expressed in the matrix form

$$\oplus^M \mathscr{B}_n(f;x) = \mathbf{f}^T(\mathbf{I} + (\mathbf{I} - \mathbf{A}) + \cdots + (\mathbf{I} - \mathbf{A})^{M-1})\,\mathbf{x}. \qquad (3.17)$$

In the limiting case, as $M \to \infty$ in (3.17), we see from Lemma 3.1 that

$$\lim_{M\to\infty} \oplus^M \mathscr{B}_n(f;x) = \mathbf{f}^T\mathbf{B}\mathbf{x}.$$

It can be easily verified by writing $[n] - [i] = q^i[n-i]$, for $i = 0, 1, \ldots, n$, that

$$\begin{bmatrix} n \\ i \end{bmatrix} \frac{1}{\lambda_i} = \frac{[n]^i}{q^{i(i-1)/2}[i]!}.$$

Now the proof follows from (3.4), Lemmas 3.1 and 3.2. ∎

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